INVESTIGATION OF TANPURA STRING VIBRATIONS USING A TWO-DIMENSIONAL TIME-DOMAIN MODEL INCORPORATING COUPLING AND BRIDGE FRICTION

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ABSTRACT

Tanpura string vibrations have been investigated previously using numerical models based on energy conserving schemes derived from a Hamiltonian description in one-dimensional form. Such time-domain models have the property that, for the lossless case, the numerical Hamiltonian (representing total energy of the system) can be proven to be constant from one time step to the next, irrespective of any of the system parameters; in practice the Hamiltonian can be shown to be conserved within machine precision. Models of this kind can reproduce a jvari effect, which results from the bridge-string interaction.

However the one-dimensional formulation has recently been shown to fail to replicate the jvaris strong dependence on the thread placement. As a first step towards simulations which accurately emulate this sensitivity to the thread placement, a two-dimensional model is proposed, incorporating coupling of controllable level between the two string polarisations at the string termination opposite from the barrier. In addition, a friction force acting when the string slides across the bridge in horizontal direction is introduced, thus effecting a further damping mechanism.

In this preliminary study, the string is terminated at the position of the thread. As in the one-dimensional model, an implicit scheme has to be used to solve the system, employing Newtons method to calculate the updated positions and momentums of each string segment.

The two-dimensional model is proven to be energy conserving when the loss parameters are set to zero, irrespective of the coupling constant. Both frequency-dependent and independent losses are then added to the string, so that the model can be compared to analogous instruments. The influence of coupling and the bridge friction are investigated.

1. INTRODUCTION

The tanpura is a traditional Indian stringed drone instrument which exhibits an interesting tonal characteristic known as the "jvari". Raman[1] theorised that the jvari (which consists of a descending formant of sustained high frequencies) also exhibited by the veena and sitar is a result of the curved bridge of these instruments. Investigation into the jvari has also been carried out by Bertrand[2] and Valette[3]. Bertrand conducted experiments using compression sensitive equipment to measure the displacement of a string vibrating against a curved barrier and was able to produce graphs showing the time varying formant phenomenon. Valette was able to predict the Helmholtz motion of the string and the precursor effect on the nut force signal due to the barrier. Despite the fact that this work has been done, a computer model which can fully simulate the behaviour of the real instrument is yet to be made, this indicates that the jvari is not yet fully understood. Current models can produce jvari like behaviour but do not capture the strong dependence of the jvari on the thread presence and placement in the tanpura. To be able to explain the jvari better more physical effects will have to be considered when attempting to describe the tanpura. This paper deals with trying to implement some previously unconsidered effects into an existing model.

When the strings are able to collide with the rigid barrier a non-linear effect is introduced. This non-linear contact effect can lead to non-physical energy jumps if the traditional Newtonian equations are used as the basis for numerical models without suitable precautions being taken [4]. These energy jumps can lead to a build up of energy which gives an unstable model. Methods which can be used to deal with this problem include energy methods and symplectic schemes. Energy preserving methods aim to conserve an energy like quantity, depending on the starting point of the model this can either lead to conserving the discrete time analogue to the total energy (Hamiltonian) [5] or an energy like quantity which does not exactly equal the true energy [6, 7]. Symplectic methods [8, 9] are ones in which the sum of all of the exterior products of the differential steps of the matched pairs of position and momentum is conserved as the system evolves over time. The already existing model which uses the Hamiltonian of the system as a basis to ensure energy conservation [10] will be briefly discussed. After this the method of expanding the model to have two oscillation planes and coupling will be detailed followed by a discussion of transverse bridge friction.

2. METHODOLOGY

An energy conserving method which preserves the Hamiltonian of the system is utilised beginning with a lossless, stiff string[11] vibrating in one dimension transverse to the string with simply supported boundary conditions. The string's dynamics can be described in the Newtonian manner:

$$\rho A \frac{\partial^2 y}{\partial t^2} = \tau \frac{\partial^2 y}{\partial z^2} - EI \frac{\partial^4 y}{\partial z^4} + k_c \lfloor (y_c - y)^\alpha \rfloor \qquad (1)$$

In this equation y is the displacement of the string, z is the distance along the string and y_c is the bridge profile as a function of z. k_c is the stiffness of the barrier, EI is the Young's modulus of the string material, α is a coefficient which defines the exponent of the force equation for the bridge repulsion, A is the cross sectional area of the string and ρ is the mass density of the string. The term $\lfloor (y_c - y)^\alpha \rfloor$ indicates that this term is zero when the string is not in contact with the barrier and has the value calculated when the string is touching the barrier. This term will always be non-negative as when the string is touching the barrier $y_c - y \geq 0$. When converted to the Hamiltonian form and discretised this gives (for a fuller description of the derivation refer to [10]):

$$H^n = c_1\big[(q^n)^t(q^n)+(y^n)^t D(y^n)+\zeta 1^t\lfloor(y_c-y^n)^{\alpha+1}\rfloor\big] \quad (2)$$

The discretisation here involves breaking the string up into M segments. H^n is the Hamiltonian over a spatial step at time step n, q^n is the scaled momentum vector at time step n of size $M-1$, y^n is the displacement vector of size $M-1$ at time step n, all y and q are at the same time step in (2), D is the spatial differentiation matrix which takes into account terms due to stiffness and tension, c_1 and ζ are constants created by combining other constants. The vectors for momentum and position are of length $M-1$ because the last point M is fixed by the boundary conditions and it is therefore never updated.

$$c_1 = \frac{2\rho A\Delta x}{\Delta t^2} \quad (3)$$

and

$$\zeta = \frac{k_c\Delta t^2}{2\rho A(\alpha+1)} \quad (4)$$

where Δx is a discretised spatial step and Δt is the times between samples. (2) can be rewritten as:

$$H^n = \Delta x\Bigg[\frac{(p^n)^t(p^n)}{2\rho A} + \frac{\tau}{2\Delta x^2}(D_1 y^n)^t(D_1(y^n)) + \frac{EI}{2\Delta x^4}(D_2(y^n))^t(D_2(y^n)) + \frac{k_c}{\alpha+1}1^t\lfloor(y_c-y^n)^{\alpha+1}\rfloor\Bigg] \quad (5)$$

so that the terms of the D matrix can be more easily interpreted. D_1 represents spatially differentiating once and D_2 twice. It can be shown[10] that the Hamiltonian is the same over two time steps, this proves that this method is energy conserving. Using Hamilton's equations of motion a scheme for finding the dynamics of the string can be constructed. After some rearranging and redefining (which can be looked up in [10]) the equation:

$$F = (I+D)s + 2(Dy^n - q^n) + \zeta s^{-1}(\lfloor(y_c-y^n-s)^{\alpha+1}\rfloor - \lfloor(y_c-y^n)^{\alpha+1}\rfloor) = 0 \quad (6)$$

can be constructed where

$$s = y^{n+1} - y^n = q^{n+1} + q^n \quad (7)$$

and I is the identity matrix. By using a Newton Raphson solver[12, 13] to converge on a value for s which solves (6) the momentum and position for each string element at the next time step can be determined. Losses are introduced by discretising their force equations and then adding them in to (6) to give:

$$F = \big[(1+\frac{\gamma\Delta t}{2})I + (1+\frac{2\eta}{\Delta t})D\big]s + 2(Dy^n - q^n) + \zeta s^{-1}(\lfloor(y_c-y^n-s)^{\alpha+1}\rfloor - \lfloor(y_c-y^n)^{\alpha+1}\rfloor) = 0 \quad (8)$$

The combined continuous domain loss equations are defined as:

$$F_{\mathrm{l}} = \eta(\tau\frac{\partial^3 y}{\partial t\partial z^2} - EI\frac{\partial^5 y}{\partial t\partial z^4}) - \rho A\gamma\frac{\partial y}{\partial t} \quad (9)$$

These losses are resistive and Kelvin-Voigt terms which come from vibration and moving through a fluid, η and γ roughly define the friction inside the string and the string going through the air. (8) is solved in the exact same way as (6) to give the update equation taking into account losses. F_l cannot cause the model to become unstable as when it is added in $\frac{\partial H}{\partial t} \leqslant 0$ [10].

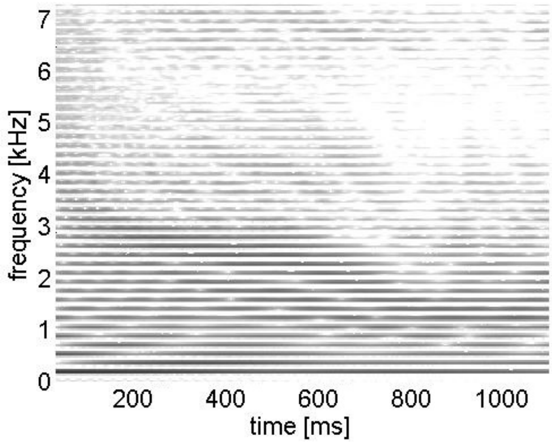

Figure 1: Spectrogram of the nut force in the y direction.

Figure 1 shows a spectrogram of the nut force in the one dimensional model, the jvari is characterised by the high frequency formant whose spectral centroid changes over time. This has the form of an intial drop in frequency followed by a plateau and then another drop. The initial condition that lead to this spectrogram was the string being set in a triangular shape with its maximum half way along the string. All simulations discussed in this paper were run with this initial condition. The constants used in the model to generate Figure 1 were; string length of $0.628\mathrm{m}$, mass per unit length of $5.5842\times10^{-4}\mathrm{Kg/m}$, tension of $31.4675\mathrm{N}$, stiffness of $8.3498\times10^{-5}\mathrm{Pam}^4$, frequency independent damping coefficient of 0.1, frequency dependent damping coefficient of 1×10^{-8}, sampling frequency of $4\times44100\mathrm{Hz}$ and 200 string segments.

3. EXPANSION TO TWO DIMENSIONS AND INTRODUCING COUPLING

When a string is plucked it is rare that it will be plucked in a manner which excites it along only one of the axis as shown in Figure 2. Even if the string was plucked in such a manner there would still be energy transferred between oscillations along each axes as the bridge and nut in a real instrument do not hold the string perfectly steady and slight asymmetries in these parts will give a transfer of energy. This can be observed on a string instrument through simple observation by plucking a string and seeing the "whirling" which takes place (whirling referring to the tendency of the string to move in a spiralling motion around the z axis as shown in Figure 2). As can be seen in Figure 2 the string will move across the bridge when whirling, this introduces a frictional force in the x direction.

To add nut coupling into a tanpura model first the model must be generalised so that the string can vibrate in the two directions. In the absence of coupling a string vibrating in two dimensions can be considered to be two separate strings as no energy can be transferred between them. This means that energy conservation (within machine precision) in the lossless case is assured as both strings conserve energy and are separate systems. As the two strings are representing two planes of oscillation of the same string they have the same physical parameters associated with them and are chosen to have the same number of string segments to keep the formulation of the new model sim-

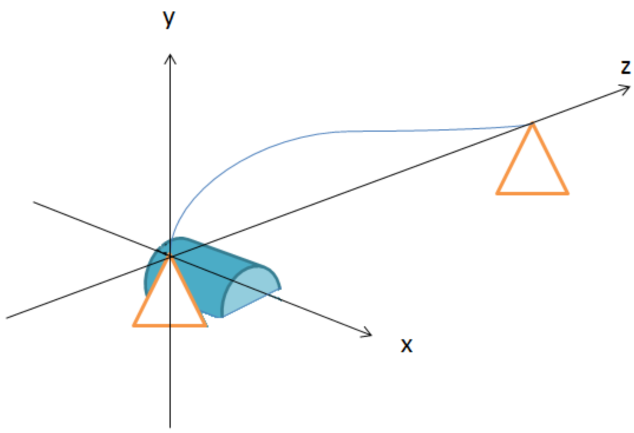

Figure 2: Diagram of the extended model. The triangles represent the points at which the string is terminated, the cylinder section the bridge and the blue line the string.

ple. The extension to two polarisations can be achieved simply though scaling up the vectors and matrices to include the other string. In (5) y^n will be replaced by:

$$r^n = \begin{pmatrix} x^n \\ y^n \end{pmatrix} \tag{10}$$

where x^n and y^n are the vectors of length $M-1$ containing all of the x and y positions of the string segments at time step n. p^n is replaced by:

$$p^n = \begin{pmatrix} {p_x}^n \\ {p_y}^n \end{pmatrix} \tag{11}$$

where ${p_x}^n$ and ${p_y}^n$ are the vectors of length $M-1$ containing all of the p_x and p_y values of the string segments at time step n. For the two dimensional string the second derivative approximation matrix is:

$$D_{2,2} = \begin{pmatrix} D_{2,1} & 0 \\ 0 & D_{2,1} \end{pmatrix} \tag{12}$$

where 0 represents a $M-1 \times M-1$ matrix of zeros and $D_{2,1}$ is the one dimensional spatial derivative matrix of the size $M-1 \times M-1$:

$$D_{2,1} = \begin{pmatrix} -2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 1 & -2 \end{pmatrix} \tag{13}$$

Using these equations in the relevant places (8) can again be used to approximate the dynamics of the system in exactly the same way as in the one dimensional case, having the effect of solving two completely strings at the same time. The barrier force now has additional zeros corresponding to the entire length of the x position vector in the relevant place as the string will not experience the repulsive barrier force in this direction. Now the update equations read:

$$r^{n+1} = s + r^n \tag{14}$$

and

$$q^{n+1} = s - q^n \tag{15}$$

where

$$s = \begin{pmatrix} s_x \\ s_y \end{pmatrix} \tag{16}$$

To introduce coupling a simple approach was adopted, the forces acting on the unfixed points closest to the nut on each string vibration axis were chosen to depend upon spatial gradients at the position of the other. There are other methods for introducing coupling such as that used by Pate[14] in the context of electric guitars. (17) shows how the second spatial derivative matrix, $D_{2,2}$, is altered with this in mind.

$$D_{2,2} = \left(\begin{array}{ccc|ccc} & & & 0 & \cdots & 0 \\ & D_{2,1} & & \vdots & \ddots & \vdots \\ & & & 0 & \cdots & \theta \\ \hline 0 & \cdots & 0 & & & \\ \vdots & \ddots & \vdots & & D_{2,1} & \\ 0 & \cdots & \theta & & & \end{array}\right) \tag{17}$$

The entries in the matrix at the points $(2M-2, M-1)$ and $(M-1, 2M-2)$ have the value θ which has the effect of enabling the transfer of energy between the strings, this can be more clearly seen by looking at the explicit equations for the second derivative approximations at these points (all y_m are at the same time point).

$$\frac{d^2 y_{M-1}}{dx^2} \approx \frac{y_{M-2} - 2y_{M-1} + \theta y_{2M-1}}{\Delta x^2} \tag{18}$$

and

$$\frac{d^2 y_{2(M-1)}}{dx^2} \approx \frac{y_{2(M-1)-1} - 2y_{2(M-1)} + \theta y_{M-1}}{\Delta x^2} \tag{19}$$

When $\theta = 1$ these equations are functionally the same as considering the system as one string. In the context of this model this results in all energy being completely transferred to the other string at the nut end point (this was also observed through testing the model). When $\theta = 0$ the strings are uncoupled and are two separate systems with no transfer of energy possible. Between 0 and 1 some energy will be transferred between the strings which is effectively coupling. θ can also lie between -1 and 0, as a wave would pass through the negative twice to return to the string where it originated the negative will cancel out. The sign of θ can be observed in the direction which the coupling force pushes the other string along its displacement axis.

When coupling is added the lossless model can be proven to still be energy conserving within machine precision in an identical way to that shown by Chatziioannou and van Walstijn[10] so long as the matrix D remains symmetric. With $\theta = 0$ the spectrogram of the nut force in the y direction would be identical to that shown in Figure 1, Figure 3 shows the spectrogram with the coupling included. Figure 3 was simulated using identical parameters (except for the introduction of the second dimension and coupling) as the simulation which generated Figure 1. A value for θ of 0.1 was assumed and used throughout, this value was chosen as it fell within the required limits and gave noticeable effects. When Figures 1 and 3 are compared some subtle differences can be observed between the spectrograms, Figure 3 has a less steep drop off at the end of the jvari and the higher frequency content above the jvari varies over time in a different manner between the two.

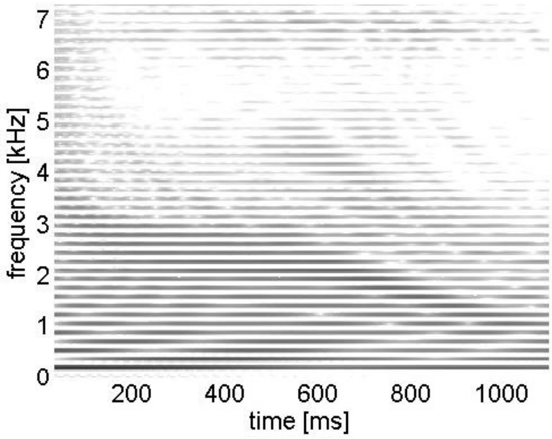

Figure 3: Spectrogram of the nut force in the y direction with coupling present.

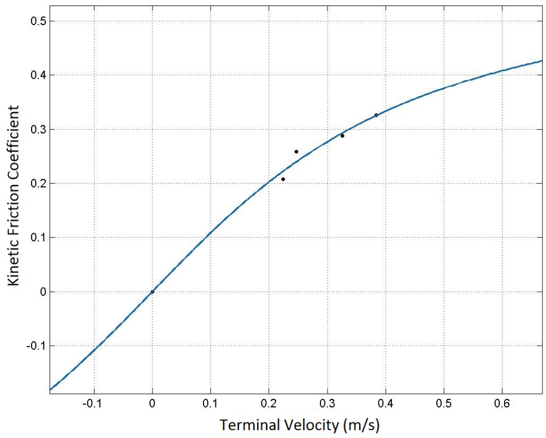

Figure 4: Graph showing the fitted atan curve (blue line)to the data gained from the experiment (black dots) detailed in section 3.3. The R-square value of the fit is 0.9913 indicating a good match.

4. BRIDGE FRICTION

4.1. Defining the tangential bridge friction and incorporating it into the model

When the string moves across the bridge in the x direction it will experience a frictional force. The form used for the kinetic friction coefficient that defines this force is a modified version of that detailed by Desvages[15], with the sticking component removed as sticking was assumed not to occur in the system being considered. This leaves the kinetic friction coefficient being defined as:

$$\mu(v) = c_1 \arctan(c_2 v) \tag{20}$$

where v is the relative velocity between the objects moving across each other, c_1 and c_2 are constants which characterise the interaction between the two materials sliding across each other. The force equation which is used in the model is:

$$F_{\mathrm{bf}} = -F_{\mathrm{n}}\mu(v_x) \tag{21}$$

The normal force, F_{n}, is the opposite of the bridge force so (20) can be rewritten in a discretised form as:

$$\begin{aligned} F_{\mathrm{bf}} = -\zeta {s_x}^{-1}(\lfloor (y_c - y^n - s_y)^{\alpha+1} \rfloor \\ - \lfloor (y_c - y^n)^{\alpha+1} \rfloor) c_1 \arctan(c_2 v_x(s_x)) \end{aligned} \tag{22}$$

The barrier friction term is added into the formulation in the same way as the other losses. The contribution of F_{bf} to F depends on both s_x because of the tangential velocity term and s_y from the barrier force term.

4.2. Finding the values of c_1 and c_2

To include the bridge friction an experiment had to be carried out to find the values of the constants in the definition of the kinetic friction as suitable data for the required materials could not be found. The experiment was carried out by allowing a tanpura bridge to slide down the three unwound strings on a guitar and taking a video of this. Unwound guitar strings were chosen as they are the most similar to tanpura string. The video was then analysed to find the velocity of the bridge over time, as the camera was only capable of taking videos at 30fps and the bridge moved quite fast there was a high uncertainty in the bridge position. Because of this uncertainty in the velocity the terminal velocity of the bridge was judged to be a more accurate measure than the velocity at each time point as it could be calculated over multiple time points. When the position against time plot became straight the linear section was fitted with a straight line and the gradient of this was taken to be the terminal velocity. Each iteration of the experiment gave a data point which consisted of a terminal velocity and the normal force. The equation which governs the bridge sliding down the guitar neck is:

$$F_{\mathrm{g}} - \mu(v) F_n = ma \tag{23}$$

where F_{g} is the gravitational force acting on the tanpura bridge, F_n is the normal force, m is the mass of the tanpura bridge and a is the acceleration of the tanpura bridge.
The normal force can be re-expressed as:

$$F_{\mathrm{n}} = F_{\mathrm{g}} \cos(\phi) \tag{24}$$

where ϕ is the angle that the guitar neck is held at compared to the horizontal axis. At terminal velocity $a = 0$ and so the equation for $\mu(v)$ can be rewritten as:

$$\mu(v) = \frac{1}{\cos(\phi)} \tag{25}$$

This allows the kinetic friction coefficient to be fitted by combining (19) and (24) into

$$c_1 \arctan(c_2 v) = \frac{1}{\cos(\phi)} \tag{26}$$

Using the angles and velocities measured as well as by noting that an additional point can be added at (0,0) because there is no kinetic friction at zero velocity the data can be fitted to an $arctan$ function using MATLAB's cftool, this is shown in Figure 4. The coefficients were determined to be $c_1 = 0.39$ and $c_2 = 2.84$. These result are reasonable (for most smooth materials the kinetic friction coefficient limit is between 0.3 and 0.7) but due to the rough nature of the experiment are estimates.

Figure 5 shows the spectrogram of the simulation run with the same parameter values as in Figure 3, apart from the addition of the barrier friction. It can be seen that the bridge friction has an effect on the shape of the simulated jvari, the second drop in the spectral centroid frequency is steeper in Figure 3.

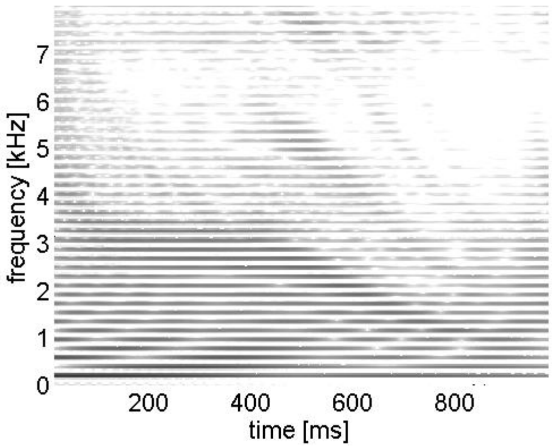

Figure 5: Simulation run with barrier friction.

5. EXAMINING THE SPECTROGRAMS OF SIMULATIONS WITH VARIED BRIDGE-THREAD DISTANCE

Spectrograms of the force in the y direction at the nut were taken to observe the jvari with different bridge centre positions. These simulations were run with the same parameters as the ones that were used to generate the other figures in this report.

It can be seen from Figure 6 that varying the bridge position has a significant effect on the shape of the jvari in the model. Changing the bridge position is analogous to varying the thread position in a real tanpura . Both involve the distance between the thread and the bridge maximum varying. Some general trends can be observed from the spectrograms, as the distance between the thread and the bridge is increased the plateau of the jvari is shortened and the central frequency of the plateau is increased. As the distance is increased the the steepness of the initial and final drops in the frequency of the jvari also increase. It can be qualitatively observed that the jvari is highly dependant on thread position in a real tanpura but experiments would have to be carried out to check whether the model correctly predicts how the jvari varies.

6. CONCLUSIONS AND FUTURE WORK

In this paper it has been shown that already existing models of the tanpura can be expanded to contain two polarisations of oscillation with coupling between them and that this new model retains the characteristic of being energy conserving to within machine precision in the lossless case. Bridge friction was also added to the model and rough estimates for necessary parameters were found by experiment. Both adding friction and coupling were shown to have an impact on the shape of the jvari but experiments will have to be carried out to ascertain a better value for θ and to check to see whether the coupling is frequency dependant. If the coupling is found to be frequency dependent then the model will have to be altered as the coupling included here is frequency independent.

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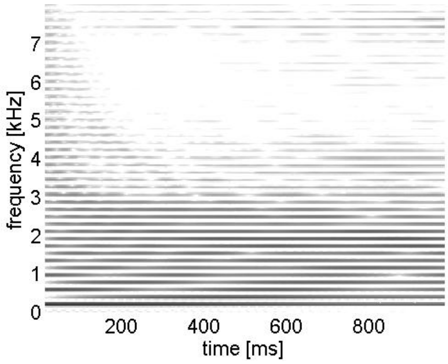
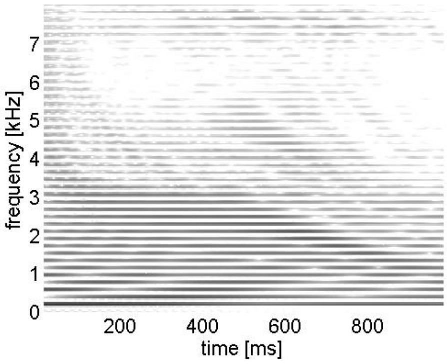
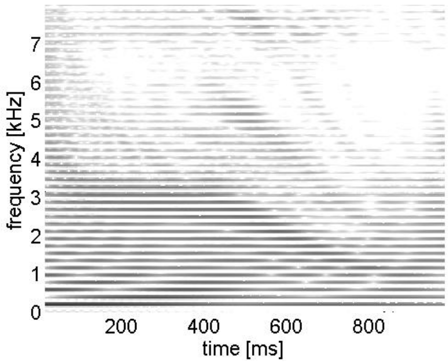


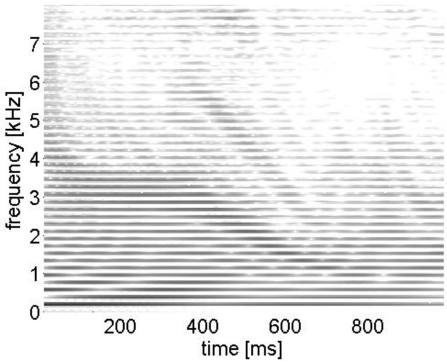
(a) distance from thread to centre of bridge 2.5mm



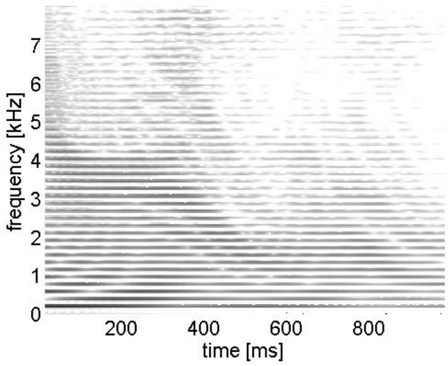
(b) distance from thread to centre of bridge 4mm



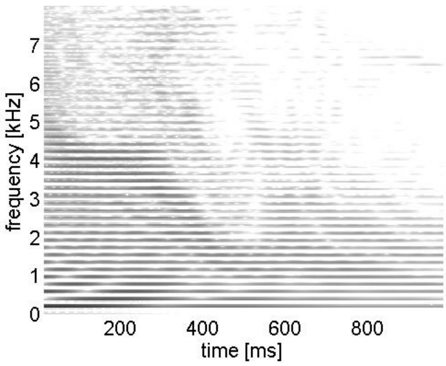
(c) distance from thread to centre of bridge 5mm



(d) distance from thread to centre of bridge 6mm



(e) distance from thread to centre of bridge 7.5mm



(f) distance from thread to centre of bridge 9mm

Figure 6: Spectrograms of simulated plucks with different bridge positions