DISCRETE-TIME CONSERVED QUANTITIES FOR DAMPED OSCILLATORS

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ABSTRACT

Numerical sound synthesis is often carried out using the finite difference time domain method. In order to analyse the stability of the derived models, energy methods can be used for both linear and nonlinear settings. For Hamiltonian systems the existence of a conserved numerical energy-like quantity can be used to guarantee the stability of the simulations. In this paper it is shown how to derive similar discrete conservation laws in cases where energy is dissipated due to friction or in the presence of an energy source due to an external force. A damped harmonic oscillator (for which an analytic solution is available) is used to present the proposed methodology. After showing how to arrive at a conserved quantity, the simulation of a nonlinear single reed shows an example of an application in the context of musical acoustics.

1. INTRODUCTION

Physics-based sound synthesis usually relies on numerical approximations of the underlying equations of motion of a vibrating system. Various time-stepping methods can be employed to this cause. In the field of Music Acoustics finite difference time domain methods provide an efficient and flexible framework for numerical simulations [1]. Of particular importance, during the formulation of such numerical schemes, is the stability of the algorithms. Stability analysis has lately been carried out using energy methods [2], which are applicable to more general systems, compared to frequency domain approaches [3]. In particular, they offer the possibility of analysing nonlinear models, whose application in musical instrument simulation has recently seen a great increase [4]. The main idea behind energy methods is for the numerical scheme to posses a conserved discrete quantity that can be used to bound the solution size. This inherently links to Hamiltonian systems (without losses), where the conserved quantity H corresponds to the (constant) system energy.

There are however a number of possible shortcomings with this approach. Firstly, for some systems dissipation plays a crucial role in the dynamics, and conservation of the internal system energy does not so readily apply. One particularly challenging example is the numerical simulation of shock waves in tubes, in which there is a need for adding further, artificial dissipation in order to avoid spurious oscillations [5]. Examples of simpler relevant sub-systems of musical acoustics interest are bow-string interaction and reed excitation of woodwinds; in both cases the oscillations are driven by a continuous energy supply. Naturally, such forced systems cannot be formulated in a stable manner without dissipative terms, and the energy is conserved only in a global sense. As shown in [1], stability of numerical schemes for the simulation of such systems can still

be defined through energy analysis, by requiring non-negativity of a numerical energy-like quantity. Nevertheless, it is logical to seek extended application of the principle of conservation in numerical modelling, which is possible by defining a more general energy-like invariant that accounts for both the dissipation from and the work injected into the modelled system. Similar ideas have been expressed in recent studies [4, 6], but explicit calculation of a numerical invariant for dissipative systems has not been demonstrated so far.

This paper explores the idea of a conserved numerical energy quantity in the presence of damping through a simple harmonic oscillator model. Using an unconditionally stable finite difference scheme as a starting point, the defined conserved quantity is tested in finite-precision realisations. The proposed methodology is first presented for the linear case and subsequently extended to include nonlinear interactions. The inclusion of a driving force is analysed leading to a numerical energy balance consisting of the system (kinetic and potential) energy, the power input due to the driving force and energy loss due to friction. The methodology is exemplified by simulation of a lumped reed (with lay beating) as a sub-system of the clarinet.

2. THE DAMPED HARMONIC OSCILLATOR

The equation of motion for the displacement y of a damped harmonic oscillator is given by

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \gamma \frac{\mathrm{d}y}{\mathrm{d}t} + \omega_0^2 y = 0 \qquad (1)$$

where γ is the damping and ω_0 the resonance frequency of the oscillator. Multiplying by the mass m of the oscillator yields

$$m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + m\gamma \frac{\mathrm{d}y}{\mathrm{d}t} + ky = 0 \qquad (2)$$

where $k = m\omega_0^2$ is the stiffness. This second order ordinary differential equation (ODE) defines an initial value problem that can be solved using either analytic or numerical methods. Note that in the case of more elaborate models, involving nonlinear interactions, an analytic solution may not be available. Two initial conditions have to be specified for this second order equation, namely $y(0)$ and $\dot{y}(0)$, where the dot signifies differentiation with respect to time. This dissipative system can be written in Hamiltonian form (see, e.g., [7]) as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial H}{\partial p} \qquad (3a)$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial y} - \gamma p \qquad (3b)$$

where H is the total energy (Hamiltonian) of the system and $p = \partial L/\partial \dot{y}$ the conjugate momentum, where L is the Lagrangian of the system [8]. Note that the specific form of (3)

using the Hamiltonian is required for obtaining a provably stable time-stepping scheme when modelling a system with non-analytic forces (which does not work when the derivative terms are first explicitly evaluated before discretisation to render the more standard first-order schemes [9]).

Defining the kinetic and potential energy as

$$T(p) = p^2/(2m), \qquad V(y) = ky^2/2 \tag{4}$$

yields

$$H(y,p) = T(p) + V(y). \tag{5}$$

The required initial condition for system (3) of first order ODE's is the pair $(y(0), p(0))$. In the undamped case (for $\gamma = 0$) this is clearly a Hamiltonian system, which conserves the total energy H [8]. In the presence of damping, energy is dissipated according to

$$
\begin{aligned}
\frac{dH}{dt} &= \frac{\partial H}{\partial y}\frac{dy}{dt} + \frac{\partial H}{\partial p}\frac{dp}{dt} = \frac{\partial H}{\partial y}\frac{p}{m} - \frac{p}{m}\left(\frac{\partial H}{\partial y} + \gamma p\right) \\
&= -\frac{\gamma p^2}{m} \leq 0
\end{aligned}
\tag{6}
$$

which induces the following conservation law

$$H + \int \frac{\gamma p^2}{m}\, dt = \text{const.} \tag{7}$$

In this paper only the case $\gamma/2 < \omega_0$ is studied. This corresponds to an underdamped system, that admits an oscillatory solution. Although higher damping values may be encountered in mechanical systems they are less relevant to sounding objects and will not be treated here. Under the above assumption for γ, taking the Laplace transform [10] of (1) yields the characteristic equation

$$s^2 + \gamma s + \omega_0^2 = 0 \tag{8}$$

which is solved by $s = \gamma/2 \pm j\,\omega_\gamma$, with $\omega_\gamma = \sqrt{\omega_0^2 - (\gamma/2)^2}$ being the frequency of the damped oscillator. The exact solution can then be written as

$$y(t) = Ae^{-\gamma t/2}\cos(\omega_\gamma t + \theta) \tag{9}$$

where the initial amplitude A and the phase θ of the oscillation can be obtained from the initial conditions.

3. DISCRETISATION

Time stepping methods involve the approximation of the continuous function $y(t)$ by a discrete function $y^n, n \geq 0$, such that $y^n \approx y(n\Delta t)$, where Δt is the time step. Using the following difference and averaging operators

$$\delta_{t+}y^n = \frac{y^{n+1} - y^n}{\Delta t}, \qquad \mu_{t+}y^n = \frac{y^{n+1} + y^n}{2} \tag{10}$$

that are centred at time $t = (n + 1/2)\Delta t$ and approximate a first order time differentiation and an identity operator respectively, equation (6), which describes the evolution of the system energy, is discretised as

$$\delta_{t+}H^n = -\frac{\gamma}{m}(\mu_{t+}p^n)^2. \tag{11}$$

By integrating (11) via summation over time from $\kappa = 0$ to n, one obtains the discrete conservation law:

$$K^n = H^{n+1} + \sum_{\kappa=0}^{n} \frac{\gamma}{m}(\mu_{t+}p^\kappa)^2\,\Delta t = \text{const.} \tag{12}$$

which is the discrete equivalent of (7).

3.1. Discretising Hamilton's equations

An energy-conserving scheme—in the sense of equation (12)—centred at time $t = (n + 1/2)\Delta t$, whose properties have been recently demonstrated for a class of nonlinear Hamiltonian systems [9], can be obtained by applying mid-point derivative approximations to (3):

$$\delta_{t+}y^n = \frac{T(p^{n+1}) - T(p^n)}{p^{n+1} - p^n} \tag{13a}$$

$$\delta_{t+}p^n = -\frac{V(y^{n+1}) - V(y^n)}{y^{n+1} - y^n} - \gamma\mu_{t+}p^n \tag{13b}$$

leading to the following (second order) numerical scheme

$$
\begin{aligned}
p^{n+1} &= \frac{1 - k\Delta t^2/4m - \gamma\Delta t/2}{1 + k\Delta t^2/4m + \gamma\Delta t/2}p^n \\
&\quad - \frac{k\Delta t}{1 + k\Delta t^2/4m + \gamma\Delta t/2}y^n
\end{aligned}
\tag{14a}
$$

$$y^{n+1} = y^n + \frac{\Delta t}{2m}(p^{n+1} + p^n). \tag{14b}$$

This scheme—which, in this linear case, is equivalent to both the trapezoidal rule and the midpoint rule—can be proven to be unconditionally stable [9] (a fact that also holds in the nonlinear case, which is not true for either the trapezoidal or the midpoint rule).

An energy relation can be derived directly from Eq. (13) by multiplying (13a) by $p^{n+1} - p^n$ and (13b) by $y^{n+1} - y^n$ and substituting by parts to get

$$H^{n+1} = H^n - \frac{\gamma}{m}\left(\mu_{t+}p^n\right)^2\Delta t \tag{15}$$

which replicates (11) exactly. Note that this derivation holds for any potential function V and hence also applies to nonlinear systems (as demonstrated in Section 4).

Simulation results for a 100 g mass with $\omega_0 = 2\pi 440$ rad/s and $\gamma = 300$ s^{-1} are compared with the analytical solution on Figure 1. The bottom plot shows the error in the conservation of the invariant quantity K, defined as $K_{\text{err}}^n = K^n - K^0$. The sampling rate for all simulations in this paper is equal to 44.1 kHz.

3.2. Driven oscillator

The energy balance defined in (7) for the continuous case and (12) for the discrete case can be extended to incorporate the action of an external driving force f_{ex}. In that case Hamilton's equations become

$$\frac{dy}{dt} = \frac{\partial H}{\partial p} \tag{16a}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial y} - \gamma p + f_{\text{ex}} \tag{16b}$$

with

$$\frac{dH}{dt} = \frac{pf_{\text{ex}} - \gamma p^2}{m} \tag{17}$$

and, discretising the external force using $\mu_{t+}f_{\text{ex}}^n$, the discrete conservation law transforms into

$$
\begin{aligned}
K^n &= H^{n+1} + \sum_{\kappa=0}^{n}\left(\gamma\left(\mu_{t+}p^\kappa\right)^2 - \left(\mu_{t+}p^\kappa\right)\left(\mu_{t+}f_{\text{ex}}^\kappa\right)\right)\frac{\Delta t}{m} \\
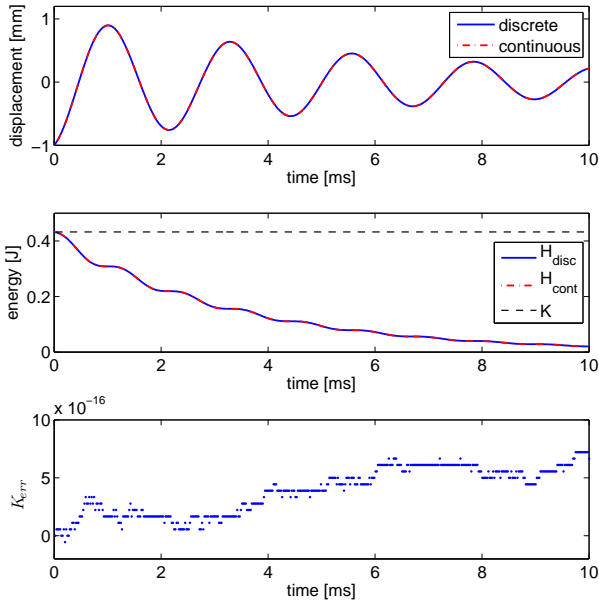&= \text{const.}
\end{aligned}
\tag{18}
$$

Figure 1: Evolution of the displacement and system energy for a damped oscillator, along with the invariant quantity. The bottom plot shows the error in the discrete conservation law (12) due to machine precision. Initial conditions are $y(0) = -1$ mm and $\dot{y}(0) = 1$ m/s.

This now accounts for both the energy dissipated due to friction and the energy fed into the system by the external force. Driving the oscillator of the previous section with a force of the form

$$f_{\text{ex}} = A \sin(\omega_{\text{ex}} t) \tag{19}$$

with $A = 1000$ N and $\omega_{\text{ex}} = 2\pi 100$ rad/s yields the results plotted in Figure 2.

4. NONLINEAR INTERACTIONS

Things become more interesting when nonlinear forces are acting on the system. In that case there is (in general) no analytic solution to the differential equation and the problem has to be treated numerically. Motivated by nonlinear interactions in musical acoustics, the nonlinear force introduced into the system is a conditional repelling force that becomes active when the displacement is greater than a certain value. The hardness of such an impact can be modelled by the contact-related stiffness k_b and a power law constant α [11] yielding

$$f_{\text{NL}} = -k_b \lfloor y^\alpha \rfloor \tag{20}$$

where $\lfloor y^\alpha \rfloor = h(y) \, y^\alpha$, $h(y)$ denoting the Heaviside step function. Thus the potential energy of the system takes the form

$$V(y) = \frac{k}{2} y^2 + \frac{k_b}{\alpha + 1} \lfloor y^{\alpha+1} \rfloor. \tag{21}$$

The Hamiltonian form of the system is the same as in (16), with this new potential V used to calculate the total energy H.

For the numerical approximation of the Hamiltonian system (16), substituting the potential energy from (21) the same procedure as in Section 3 can be used, using mid-point derivative approximations to discretise Hamilton's equations at time
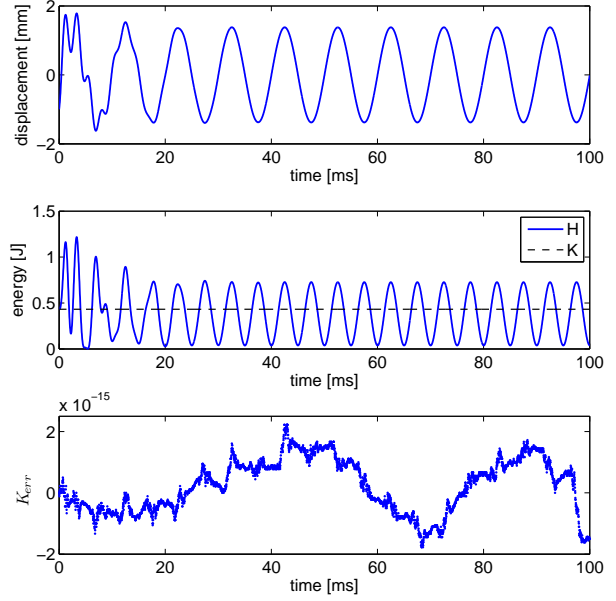

Figure 2: Evolution of the displacement and system energy for a driven oscillator, along with the invariant quantity. The bottom plot shows the error in the discrete conservation law (18) due to machine precision. Initial conditions are as in Fig. 1.

$t = (n + 1/2)\Delta t$. This leads to the following nonlinear equation

$$F(x) = \left(1 + \frac{\gamma \Delta t}{2}\right) x + \frac{\Delta t^2}{2m} \frac{V(x + y^n) - V(y^n)}{x}$$
$$- \frac{\Delta t}{m} p^n - \frac{\Delta t^2}{2m} (\mu_{t+} f_{\text{ex}}) = 0 \tag{22}$$

with $x = y^{n+1} - y^n = (\Delta t/2m)(p^{n+1} + p^n)$. Thus solving for x the scheme can be updated in y and p. The nonlinear equation (22) can be shown to have a unique solution (see [12]) and the conserved quantity is the same as in Section 3.2, now taking into account the updated potential energy from (21) when calculating H.

4.1. Application to sound synthesis

The methodology presented above for discretising nonlinear systems and monitoring their energy balance can be applied to several settings in sound synthesis simulations. In this case, it is assumed that the external driving force is known. In practise f_{ex} may be due to interaction between different objects, or supplied real-time by a performer. Applying a periodic driving force usually results in a steady-state displacement signal preceded by a transient oscillation, like that observed in Figure 2.

For instance, if the pressure difference that drives a clarinet reed is considered given in the form of a time series $p_\Delta(t)$, it is possible to sample it and calculate the force driving the reed, in order to simulate its oscillations. Defining M as the mass per unit area of the reed, the equation of motion becomes

$$M \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + M\gamma \frac{\mathrm{d}y}{\mathrm{d}t} + M\omega_0^2 y + k_c \lfloor (y - y_c)^\alpha \rfloor = p_\Delta \tag{23}$$

where k_c is defined as contact stiffness per unit area and y_c is the point after which the reed-mouthpiece interaction becomes significant [13]. The driving force per unit area corresponds to the pressure difference across the reed. The results of such a
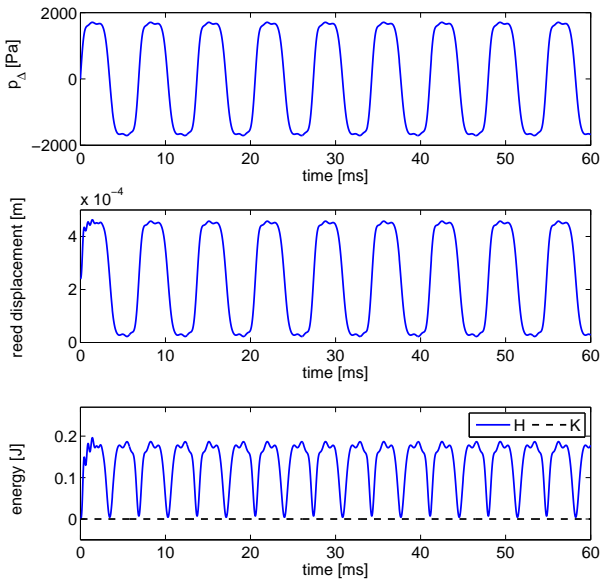
Figure 3: Evolution of the displacement and energy for a single reed, driven by a clarinet-like pressure difference p_Δ (as shown on the top plot). Initial conditions are $y(0) = 0$ and $\dot{y}(0) = 0$.

Table 1: Parameters used in the single-reed simulation.

parameter	value	unit
ω_0	$2\pi 2000$	rad/s
γ	3000	1/s
y_c	2.4×10^{-4}	m
M	0.05	kg/m^2

simulation, implemented by solving equation (22) at each time step, are shown in Figure 3. The pressure difference p_Δ is synthesised taking into account a typical clarinet spectrum at 146 Hz (Note D3) [14], by defining the amplitudes of the first seven harmonics as

$$\{A_1, A_2, \ldots A_7\} = \{2000, 40, 400, 40, 100, 40, 28\} \text{ Pa.}$$

The contact is modelled using $k_c = 8.23 \times 10^{10}$ and $\alpha = 2$ and the reed parameters are given on Table 1. It can be observed that the conserved quantity K remains constant, in accordance with equation (18), which shows that no 'artificial' energy is fed into or lost from the system, besides that due to the external driving and frictional forces. Figure 4 plots the error in the conservation of K^n, defined as $K_{\text{err}} = K^{n+1} - K^n$. This demonstrates that in practical simulations, the global energy is not exactly preserved per time step, but varies by a small (quantised) amount due to finite precision; as explained in [1], such plots are also a useful debugging tool.
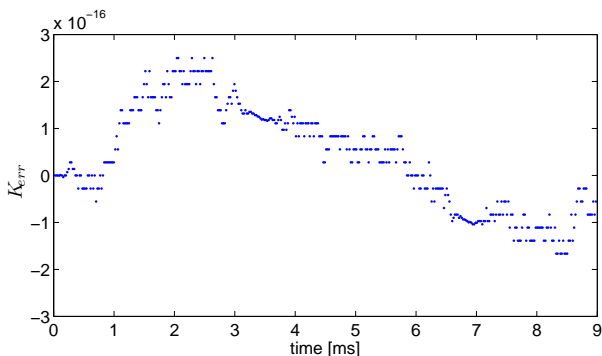

Figure 4: Evolution of the energy error K_{err} during the first 9 ms of the clarinet reed simulation.

5. DISCUSSION

Conservation of energy has been recognised as a key principle within stability analysis of time-stepping algorithms for a few decades [15]; more recently it has emerged as such in the context of simulation of non-linear phenomena in musical instruments [4]. Such an approach has evolved within the consideration of lossless systems, where the energy of the system is indeed conserved. A simple extension has been presented here, where a similar conservation law is formulated for damped oscillators. Conserved quantities, involving the damping parameter γ and the external driving force f_{ex}, have been calculated for a discretisation approach that is known to be unconditionally stable. It has been verified that the error in the conservation of these quantities is within the order of machine accuracy.

Such a test is an additional tool to verify the validity of a numerical approximation. Frictional forces lead to a monotonically decreasing energy and as such can conceal a potential instability of a numerical simulation. Even when conserved quantities can be theoretically established, in practice their conservation may depend on the range of the model parameters [9]. Testing for such an energy drift is usually carried out in conservative systems, by monitoring the constant system energy. The conserved quantity K proposed here can serve as a similar tool in the analysis of damped systems.

6. REFERENCES

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